

A SHORT PROOF OF A CONJECTURE BY HIRSCHHORN AND SELLERS ON OVERPARTITIONS

LIUQUAN WANG

ABSTRACT. Let $\overline{p}(n)$ be the number of overpartitions of n , we establish and give a short elementary proof of the following congruence

$$\overline{p}(4^\alpha(40n + 35)) \equiv 0 \pmod{40},$$

where α, n are nonnegative integers. By letting $\alpha = 0$ we proved a conjecture of Hirschhorn and Sellers. Some new congruences for $\overline{p}(n)$ modulo 3 and 5 have also been found, including the following two infinite families of Ramanujan-type congruences: for any integers $n \geq 0$ and $\alpha \geq 1$,

$$\overline{p}(5^{2\alpha+1}(5n + 1)) \equiv \overline{p}(5^{2\alpha+1}(5n + 4)) \equiv 0 \pmod{5}.$$

1. INTRODUCTION AND MAIN RESULTS

An overpartition of an integer n is a partition wherein the first occurrence of a part may be overlined, and the number of overpartitions of n is denoted by $\overline{p}(n)$. It's well known (see [4], for example) that the generating function for $\overline{p}(n)$ is

$$(1) \quad \sum_{n \geq 0} \overline{p}(n)q^n = \frac{(-q; q)_\infty}{(q; q)_\infty} = \frac{1}{\varphi(-q)},$$

where $(a; q)_\infty = (1 - a)(1 - aq) \cdots (1 - aq^n) \cdots$ is a standard q series notation, and $\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$ is one of Ramanujan's Theta functions.

Overpartition was first introduced by MacMahon [13] and has received much attention during the past ten years. There're numerous results concerning the arithmetic properties of $\overline{p}(n)$, for more information and references, we refer the reader to see [2], [4] and [7] - [14]. Here we only mention some results which are related to our work.

In 2005, Hirschhorn and Sellers [8] gave many Ramanujan-type identities about $\overline{p}(n)$, for example

$$(2) \quad \sum_{n \geq 0} \overline{p}(4n + 3) = 8 \frac{(q^2; q^2)_\infty (q^4; q^4)_\infty^6}{(q; q)_\infty^8},$$

which clearly implies $\overline{p}(4n+3) \equiv 0 \pmod{8}$. Meanwhile, they proposed a curious conjecture:

Conjecture 1. *For any integer $n \geq 0$, we have*

$$\overline{p}(40n + 35) \equiv 0 \pmod{40}.$$

In a recent paper, Chen and Xia [2] give a proof of Conjecture 1 by employing 2-dissections of quotients of theta functions and (p, k) -parametrization of theta functions. Their proof is relatively long and complicate, and our main goal is to give a short proof. Indeed, let $r_k(n)$ denotes the number of representations of n as sum of k squares, we find the following arithmetic relation:

Date: April 20, 2014.

2010 Mathematics Subject Classification. Primary 05A17; Secondary 11P83.

Key words and phrases. overpartition, congruence, sum of squares.

Theorem 1. *For any integer $n \geq 1$, we have*

$$\overline{p}(5n) \equiv (-1)^n r_3(n) \pmod{5}.$$

We have two remarkable corollaries.

Corollary 1. *For any integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\overline{p}(4^\alpha(40n + 35)) \equiv 0 \pmod{5},$$

and

$$\overline{p}(5 \cdot 4^{\alpha+1}n) \equiv (-1)^n \overline{p}(5n) \pmod{5}.$$

Let $\alpha = 0$ in Corollary 1, we get $\overline{p}(40n + 35) \equiv 0 \pmod{5}$, combining this with (2), Conjecture 1 follows immediately.

Corollary 2. *For any prime $p \equiv -1 \pmod{5}$, we have*

$$\overline{p}(5p^3n) \equiv 0 \pmod{5}$$

for all n coprime to p .

Corollary 2 was first proved by Treneer (see Proposition 1.4 in [16]) in 2006 using the theory of modular forms, which is not elementary.

Furthermore, with Corollary 1 in mind, we are able to generalize Conjecture 1 to the following

Theorem 2. *For any integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\overline{p}(4^\alpha(40n + 35)) \equiv 0 \pmod{40}.$$

Some miscellaneous congruences can be deduced from Theorem 1, we list some of them here.

Theorem 3. *For any integers $\alpha \geq 1$ and $n \geq 0$, we have*

$$\overline{p}(5^{2\alpha+1}(5n + 1)) \equiv \overline{p}(5^{2\alpha+1}(5n + 4)) \equiv 0 \pmod{5}.$$

Theorem 4. *Let $p \geq 3$ be a prime, N a positive integer which is coprime to p . Let α be any nonnegative integer.*

- (1) *If $p \equiv 1 \pmod{5}$, then $\overline{p}(5p^{10\alpha+9}N) \equiv 0 \pmod{5}$.*
- (2) *If $p \equiv 2, 3, 4 \pmod{5}$, then $\overline{p}(5p^{8\alpha+7}N) \equiv 0 \pmod{5}$.*

Finally, we mention that in 2011, based on the generating function of $\overline{p}(3n)$ discovered by Hirschhorn and Sellers [7], Lovejoy and Osburn [12] proved: for any integer $n \geq 1$, we have

$$\overline{p}(3n) \equiv (-1)^n r_5(n) \pmod{3}.$$

Using the same method from the proof of Theorem 1, we are able to improve this congruence to the following one.

Theorem 5. *For any integer $n \geq 1$, we have*

$$\overline{p}(3n) \equiv (-1)^n r_5(n) \pmod{9}.$$

Similar to Theorem 4, we can deduce the following interesting congruences from Theorem 5.

Theorem 6. *Let $p \geq 3$ be a prime and N a positive integer which is coprime to p .*

- (1) *If $p \equiv 1 \pmod{3}$, then $\overline{p}(3p^{6\alpha+5}N) \equiv 0 \pmod{3}$, and $\overline{p}(3p^{18\alpha+17}N) \equiv 0 \pmod{9}$.*
- (2) *If $p \equiv 2 \pmod{3}$, then $\overline{p}(3p^{4\alpha+3}N) \equiv 0 \pmod{9}$.*

2. PRELIMINARIES

Lemma 1 (cf. Lemma 1.2 in [15]). *Let p be a prime and α a positive integer. Then*

$$(q; q)_\infty^{p^\alpha} \equiv (q^p; q^p)_\infty^{p^{\alpha-1}} \pmod{p^\alpha}.$$

Lemma 2 (cf. Theorem 3.3.1, Theorem 3.5.4 in [1]). *For any integer $n \geq 1$, we have*

$$r_4(n) = 8 \sum_{d|n, 4 \nmid d} d, \quad r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3.$$

Lemma 3. *For any prime $p \geq 3$, we have*

$$r_4(pn) \equiv r_4(n) \pmod{p}, \quad r_8(pn) \equiv r_8(n) \pmod{p^3}.$$

Proof. By Lemma 2, we have

$$r_4(n) = 8 \sum_{d|n, 4 \nmid d} d = 8 \sum_{\substack{d|n \\ 4 \nmid d, p \nmid d}} d + 8 \sum_{\substack{d|n \\ 4 \nmid d, p|d}} d \equiv 8 \sum_{\substack{d|n \\ 4 \nmid d, p \nmid d}} d \pmod{p},$$

and

$$r_4(pn) = 8 \sum_{d|pn, 4 \nmid d} d = 8 \sum_{\substack{d|pn \\ 4 \nmid d, p \nmid d}} d + 8 \sum_{\substack{d|pn \\ 4 \nmid d, p|d}} d = 8 \sum_{\substack{d|n \\ 4 \nmid d, p \nmid d}} d + 8p \sum_{\substack{d|n \\ 4 \nmid d}} d.$$

Comparing the two identities above, we see that $r_4(pn) \equiv r_4(n) \pmod{p}$.

Similarly we can prove $r_8(pn) \equiv r_8(n) \pmod{p^3}$. \square

Lemma 4 (cf. Theorem 1 in Chapter 4 of [5]). *For any integers $\alpha \geq 0$ and $n \geq 0$, we have $r_3(4^\alpha(8n+7)) = 0$ and $r_3(4^\alpha n) = r_3(n)$.*

Lemma 5 (cf. [6]). *Let $p \geq 3$ be a prime, for any integers $n \geq 1$ and $\alpha \geq 0$, we have*

$$r_3(p^{2\alpha}n) = \left(\frac{p^{\alpha+1}-1}{p-1} - \left(\frac{-n}{p} \right) \frac{p^\alpha-1}{p-1} \right) r_3(n) - p \frac{p^\alpha-1}{p-1} r_3(n/p^2).$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol, and we take $r_3(n/p^2) = 0$ unless $p^2|n$.

Lemma 6 (cf. Theorem 3 in [9]). *Let n be an integer which is neither a square nor twice a square, then $\overline{p}(n) \equiv 0 \pmod{8}$.*

Lemma 7 (cf. [3]). *Let $p \geq 3$ be a prime, n a positive integer and $p^2 \nmid n$. For any integer $\alpha \geq 0$, we have*

$$r_5(p^{2\alpha}n) = \left(\frac{p^{3\alpha+3}-1}{p^3-1} - p \left(\frac{n}{p} \right) \frac{p^{3\alpha}-1}{p^3-1} \right) r_5(n).$$

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. Replace q by $-q$ in (1) we get

$$\sum_{n \geq 0} \overline{p}(n)(-q)^n = \frac{1}{\varphi(q)},$$

hence we have

$$\varphi(q)^5 \sum_{n \geq 0} \overline{p}(n)(-q)^n = \varphi(q)^4 = \sum_{n \geq 0} r_4(n)q^n.$$

By Lemma 1, we have $\varphi(q)^5 \equiv \varphi(q^5) \pmod{5}$ and thus

$$\varphi(q^5) \sum_{n \geq 0} \overline{p}(n)(-q)^n \equiv \sum_{n \geq 0} r_4(n)q^n \pmod{5}.$$

Collecting all the terms of the form q^{5n} on both sides, we get

$$\varphi(q^5) \sum_{n \geq 0} \bar{p}(5n)(-q)^{5n} \equiv \sum_{n \geq 0} r_4(5n)q^{5n} \pmod{5},$$

replace q^5 by q and apply Lemma 3 with $p = 5$ we obtain

$$\varphi(q) \sum_{n \geq 0} \bar{p}(5n)(-q)^n \equiv \sum_{n \geq 0} r_4(5n)q^n \equiv \sum_{n \geq 0} r_4(n)q^n = \varphi(q)^4 \pmod{5}.$$

Hence we have

$$\sum_{n \geq 0} \bar{p}(5n)(-q)^n \equiv \varphi(q)^3 = \sum_{n \geq 0} r_3(n)q^n \pmod{5},$$

and Theorem 1 follows by comparing the coefficients of q^n on both sides. \square

Proof of Corollary 1. This corollary follows immediately by Theorem 1 and Lemma 4. \square

Proof of Corollary 2. By Theorem 1, we can replace $r_3(n)$ by $(-1)^n \bar{p}(5n)$ throughout Lemma 5 with $\alpha = 1$, we deduce that

$$\bar{p}(5p^2m) \equiv \left(p + 1 - \left(\frac{-m}{p}\right)\right) \bar{p}(m) - p \bar{p}(m/p^2) \pmod{5}.$$

Let $m = np$, then $\bar{p}(m/p^2) = \bar{p}(n/p) = 0$ since n is coprime to p , the theorem follows immediately. \square

Proof of Theorem 2. Since $40n + 35 = 5(8n + 7)$ is an odd number, it can not be twice a square. If $5(8n + 7) = x^2$ is a square where x is an odd number, then we know $5|x$. Let $x = 5y$ where y is an odd number, we get $8n + 7 = 5y^2$, but $5y^2 \equiv 5 \pmod{8}$, this is a contradiction. Hence we know $4^\alpha(40n + 35)$ is neither a square nor twice a square, by Lemma 6 we have $\bar{p}(4^\alpha(40n + 35)) \equiv 0 \pmod{8}$, combining this with Corollary 1 we complete our proof. \square

Proof of Theorem 3. Set $p = 5$ in Lemma 5, $n = 5m + r, r \in \{1, 4\}$, it's easy to deduce $r_3(5^{2\alpha}(5m + r)) \equiv 0 \pmod{5}$ for any integer $\alpha \geq 1$. By Theorem 1 we complete our proof. \square

Proof of Theorem 4. (1) Let $n = pN$ in Lemma 5, then replace α by $5\alpha + 4$, we have

$$\frac{p^{5\alpha+5} - 1}{p - 1} = 1 + p + \cdots + p^{5\alpha+4} \equiv 0 \pmod{5},$$

hence $r_3(p^{10\alpha+9}N) \equiv 0 \pmod{5}$. By Theorem 1 we deduce $\bar{p}(5p^{10\alpha+9}N) \equiv 0 \pmod{5}$.

(2) Let $n = pN$ in Lemma 5, then replace α by $4\alpha + 3$, since $p^{4\alpha+4} \equiv 1 \pmod{5}$, we deduce $r_3(p^{8\alpha+7}N) \equiv 0 \pmod{5}$, by Theorem 1 we deduce $\bar{p}(5p^{8\alpha+7}N) \equiv 0 \pmod{5}$. \square

Proof of Theorem 5. We have

$$\varphi(q)^9 \sum_{n \geq 0} \bar{p}(n)(-q)^n = \varphi(q)^8 = \sum_{n \geq 0} r_8(n)q^n.$$

Thanks to Lemma 1, we have $\varphi(q)^9 \equiv \varphi(q^3)^3 \pmod{9}$ and thus

$$\varphi(q^3)^3 \sum_{n \geq 0} \bar{p}(n)(-q)^n \equiv \sum_{n \geq 0} r_8(n)q^n \pmod{9}.$$

Collecting all the terms of the form q^{3n} on both sides, we get

$$\varphi(q^3)^3 \sum_{n \geq 0} \bar{p}(3n)(-q)^{3n} \equiv \sum_{n \geq 0} r_8(3n)q^{3n} \pmod{9},$$

replace q^3 by q and apply Lemma 3 with $p = 3$, we obtain

$$\varphi(q)^3 \sum_{n \geq 0} \bar{p}(3n)(-q)^n \equiv \sum_{n \geq 0} r_8(3n)q^n \equiv \sum_{n \geq 0} r_8(n)q^n = \varphi(q)^8 \pmod{9}.$$

Hence we have

$$\sum_{n \geq 0} \bar{p}(3n)(-q)^n \equiv \varphi(q)^5 = \sum_{n \geq 0} r_5(n)q^n \pmod{9},$$

and Theorem 5 follows by comparing the coefficients of q^n on both sides. \square

Proof of Theorem 6. (1) Let $n = pN$ in Lemma 7, then replace α by $3\alpha + 2$, we have

$$\frac{p^{9\alpha+9} - 1}{p^3 - 1} = 1 + p^3 + \cdots + p^{3(3\alpha+2)} \equiv 0 \pmod{3},$$

hence $r_5(p^{6\alpha+5}N) \equiv 0 \pmod{3}$. By Theorem 5 we deduce $\bar{p}(3p^{6\alpha+5}N) \equiv 0 \pmod{3}$.

Similarly, let $n = pN$ in Lemma 7 and replace α by $9\alpha + 8$. Since $p \equiv 1 \pmod{3}$ implies $p^3 \equiv 1 \pmod{9}$, we have

$$\frac{p^{27\alpha+27} - 1}{p^3 - 1} = 1 + p^3 + \cdots + p^{3(9\alpha+8)} \equiv 0 \pmod{9}.$$

Hence $r_5(p^{18\alpha+17}N) \equiv 0 \pmod{9}$, by Theorem 5 we deduce $\bar{p}(3p^{18\alpha+17}N) \equiv 0 \pmod{9}$.

(2) Let $n = pN$ in Lemma 7, then replace α by $2\alpha + 1$. Note that $p \equiv 2 \pmod{3}$ implies $p^3 \equiv -1 \pmod{9}$. Since $p^{6\alpha+6} \equiv 1 \pmod{9}$, we have $r_5(p^{4\alpha+3}N) \equiv 0 \pmod{9}$. By Theorem 5 we deduce $\bar{p}(3p^{4\alpha+3}N) \equiv 0 \pmod{9}$. \square

REFERENCES

1. B. C. Berndt, Number Theory in the Spirit of Ramanujan, Am. Math. Soc., Providence, 2006.
2. W. Y. C. Chen, E. X. W. Xia, Proof of a conjecture of Hirschhorn and Sellers on overpartitions, Acta Arith., to appear.
3. S. Cooper, Sums of Five, Seven and Nine Squares, Ramanujan J., 6 (2002), 469-490.
4. S. Corteel and J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc. 356 (2004), 1623-1635.
5. E. Grosswald, Representations of Integers as Sums of Squares, Springer-Verlag, New York, 1985.
6. M. D. Hirschhorn, J. A. Sellers, On representations of a number as sum of three squares, Discrete Math. 199 (1999), 85-101.
7. M. D. Hirschhorn, J. A. Sellers, An infinite family of overpartition congruences modulo 12, Integers 5 (2005), Article A20.
8. M. D. Hirschhorn, J. A. Sellers, Arithmetic relations for overpartitions, J. Combin. Math. Combin. Comput. 53 (2005), 65-73.
9. B. Kim, A short note on the overpartition function, Discrete Math. 309 (2009), 2528-2532.
10. B. Kim, The overpartition function modulo 128, Integers 8 (2008), Article A38.
11. J. Lovejoy, Overpartition theorems of the Rogers-Ramanujan type, J. London Math. Soc. 69 (2004), 562-574.
12. J. Lovejoy, R. Osburn, Quadratic forms and four partition functions modulo 3, Integers 11 (2011).
13. P. A. MacMahon, Combinatory Analysis, Chelsea, New York (1960).
14. K. Mahlburg, The overpartition function modulo small powers of 2, Discrete Math. 286 (3) (2004), 263-267.
15. S. Radu, J. A. Sellers, Congruence properties modulo 5 and 7 for the pod function, Int. J. Number Theory, 07 (2011), 2249.
16. S. Treneer, Congruences for the coefficients of weakly holomorphic modular forms, Proc. London Math. Soc. 93 (2006), 304-324.

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, SINGAPORE, 119076, SINGAPORE
E-mail address: mathlqwang@163.com